29

Answers and Explanations to BC Practice Exam 1

ANSWERS AND EXPLANATIONS TO SECTION I

PROBLEM 1. If $7 = xy - e^{xy}$, then $\dfrac{dy}{dx} =$

We need to use Implicit Differentiation to solve this problem.

Step 1: $0 = \left(x\dfrac{dy}{dx} + y\dfrac{dx}{dx} \right) - \left[\left(x\dfrac{dy}{dx} + y\dfrac{dx}{dx} \right)e^{xy} \right]$ *Remember:* $\dfrac{dx}{dx} = 1$

Step 2: $0 = \left(x\dfrac{dy}{dx} + y \right) - \left[\left(x\dfrac{dy}{dx} + y \right)e^{xy} \right]$

Step 3: $0 = \left(x\dfrac{dy}{dx} + y \right) - xe^{xy}\dfrac{dy}{dx} - ye^{xy}$

Step 4: $ye^{xy} - y = x\dfrac{dy}{dx} - xe^{xy}\dfrac{dy}{dx}$

Step 5: $ye^{xy} - y = \dfrac{dy}{dx}\left(x - xe^{xy} \right)$

Step 6: $\dfrac{\left(ye^{xy} - y \right)}{\left(x - xe^{xy} \right)} = \dfrac{dy}{dx}$

Step 7: $\dfrac{\left(ye^{xy} - y \right)}{\left(x - xe^{xy} \right)} = \dfrac{y\left(e^{xy} - 1 \right)}{x\left(1 - e^{xy} \right)} = \dfrac{-y}{x}$

The answer is (D).

PROBLEM 2. The volume of the solid that results when the area between the curve $y = e^x$ and the line $y = 0$, from $x = 1$ to $x = 2$, is revolved around the *x-axis* is

Step 1: We need to use the formula for finding the Volume of a Solid of Revolution. The equations in the problems are given to us in terms of x, and we are rotating around the *x-axis*, so we can use the method of washers. The curve $y = e^x$ is always above the curve $y = 0$ (which is the *x-axis*), so we don't have to break this up into two integrals.
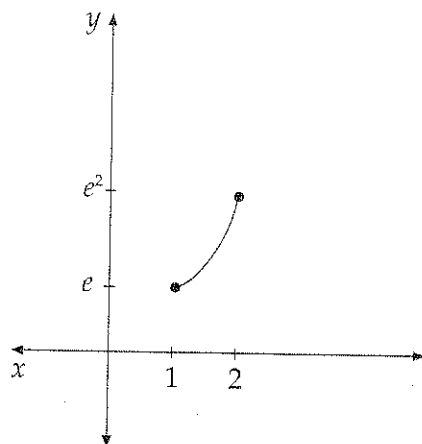
Therefore, the integral will be: $\pi \int_1^2 \left(\left(e^x\right)^2 - 0^2 \right) dx$

Step 2: $\pi \int_1^2 \left(\left(e^x\right)^2 - 0^2 \right) dx = \pi \int_1^2 e^{2x} dx$

Step 3: $\pi \int_1^2 e^{2x} dx = \pi \left(\frac{1}{2} e^{2x} \right) \Big|_1^2$

Step 4: $\pi \left(\frac{1}{2} e^{2x} \right) \Big|_1^2 = \pi \left(\frac{1}{2} e^4 - \frac{1}{2} e^2 \right)$

Step 5: $\pi \left(\frac{1}{2} e^4 - \frac{1}{2} e^2 \right) = \frac{\pi}{2} \left(e^4 - e^2 \right)$

The answer is (B).

PROBLEM 3. $\displaystyle \int \frac{x-18}{(x+3)(x-4)} dx =$

Step 1: We need to use partial fractions to evaluate this integral.

First, we write the integrand as: $\displaystyle \frac{x-18}{(x+3)(x-4)} = \frac{A}{(x+3)} + \frac{B}{(x-4)}$

Step 2: Multiply both sides by: $(x + 3)(x - 4)$

Step 3: Now we have: $x - 18 = A(x - 4) + B(x + 3)$

Step 4: $x - 18 = Ax - 4A + Bx + 3B$

Step 5: $x - 18 = (Ax + Bx) + (3B - 4A)$

Step 6: $x - 18 = x(A + B) + (3B - 4A)$

Step 7: Thus: $(A+B)=1$ and $(3B-4A)=-18$

Step 8: Solve this using simultaneous equations to get: $A = 3$ and $B = -2$

Step 9: We can now rewrite the original integral as: $\displaystyle \int\left(\frac{3}{x+3}+\frac{-2}{x-4}\right)dx$

Step 10: Which is the same as: $\displaystyle \int\frac{3dx}{x+3}-\int\frac{2dx}{x-4}$

The answer is (E).

PROBLEM 4. If $y = 5x^2 + 4x$ and $x = \ln t$ then $\dfrac{dy}{dt} =$

We can solve this problem with the Chain Rule.

Step 1: $\dfrac{dy}{dx}=10x+4$

Step 2: $\dfrac{dx}{dt}=\dfrac{1}{t}$

Step 3: $\dfrac{dy}{dt}=\dfrac{dy}{dx}\dfrac{dx}{dt}$ so $\dfrac{dy}{dt}=(10x+4)\left(\dfrac{1}{t}\right)$

Step 4: Substitute for x so that $\dfrac{dy}{dt}=(10\ln t+4)\left(\dfrac{1}{t}\right)=\dfrac{10\ln t+4}{t}$

The answer is (C).

You could also have solved this by first substituting for x in the original equation and getting y in terms of t, and then differentiating with the chain rule.

PROBLEM 5. $\displaystyle \int_{0}^{\frac{\pi}{2}} \sin^5 x\cos x\,dx =$

This Trigonometric integral is solved by using u-substitution.

Step 1: Let $\begin{aligned} u &= \sin x \\ du &= \cos x\,dx \end{aligned}$

Step 2: The integral is now written as: $\int_0^1 u^5 du =$

Note: The limits of integration change because

$$u = \sin\frac{\pi}{2} = 1 \text{ and } u = \sin 0 = 0$$

Step 3: $\int_0^1 u^5 du = \frac{u^6}{6}\Big|_0^1$

Step 4: $\frac{u^6}{6}\Big|_0^1 = \frac{1}{6} - 0 = \frac{1}{6}$

The answer is (A).

PROBLEM 6. The tangent line to the curve $y = x^3 - 4x + 8$ at the point (2, 8) has an x-intercept at

Step 1: First find the slope of the tangent line. $\frac{dy}{dx} = 3x^2 - 4$

Step 2: Plug x into $\frac{dy}{dx} = 3(2)^2 - 4 = 12 - 4 = 8$.

This means that the slope of the tangent line at $x=2$ is 8.

Step 3: Then the equation of the tangent line is $(y - 8) = 8(x - 2)$.

Step 4: The x-intercept is found by Plugging In $y=0$ and solving for x.

$$0 - 8 = 8(x - 2) \quad \text{Therefore } x = 1.$$

The answer is (B).

PROBLEM 7. The graph in the xy-plane represented by $x = 3 \sin(t)$ and $y = 2\cos(t)$ is

This is a Parametric Equation and is solved by eliminating t from the equations and finding a direct relationship between y and x. These problems can often be quite difficult, but, fortunately, on the AP, they only give very easy versions of parametric equations.

Step 1: $\frac{x}{3} = \sin(t)$ and $\frac{y}{2} = \cos(t)$

Step 2: Because $\sin^2(t) + \cos^2(t) = 1$, we can substitute and we get $\left(\dfrac{x}{3}\right)^2 + \left(\dfrac{y}{2}\right)^2 = 1$.

This is an ellipse.

The answer is (B).

PROBLEM 8. $\displaystyle\int \dfrac{dx}{\sqrt{4-9x^2}} =$

You should recognize this as an Inverse Trigonometric Integral of the form $\dfrac{du}{\sqrt{1-u^2}}$, which is $\sin^{-1}(u)$. You also should know this by looking at the answer choices. One of the difficulties of the AP is that you are required to recognize many different types of integrals by sight, and to know which techniques to use to solve them.

Step 1: First, we need to use a little algebra to convert the integrand to the form $\dfrac{du}{\sqrt{1-u^2}}$

Rewrite $\sqrt{\left(4-9x^2\right)} = \sqrt{4\left(1-\dfrac{9x^2}{4}\right)} = 2\sqrt{\left(1-\dfrac{9x^2}{4}\right)} = 2\sqrt{1-\left(\dfrac{3x}{2}\right)^2}$

The integral then becomes: $\displaystyle\int \dfrac{dx}{\sqrt{4-9x^2}} = \int \dfrac{dx}{\left(2\sqrt{1-\left(\dfrac{3x}{2}\right)^2}\right)} = \dfrac{1}{2}\int \dfrac{dx}{\sqrt{1-\left(\dfrac{3x}{2}\right)^2}}$

Step 2: Now we can use **u-substitution.** Let $u = \dfrac{3x}{2}$ and $du = \dfrac{3}{2}dx$ and $\dfrac{2}{3}du = dx$

We can now rewrite the integral as: $\displaystyle\int \dfrac{dx}{\sqrt{4-9x^2}} = \left(\dfrac{1}{2}\right)\left(\dfrac{2}{3}\right)\int \dfrac{du}{\sqrt{1-u^2}} = \dfrac{1}{3}\int \dfrac{du}{\sqrt{1-u^2}}$

Step 3: You should have memorized this last integral. It is $\left(\dfrac{1}{3}\right)\sin^{-1}(u) + C$.

Step 4: Now reverse the u-substitution and we have:

$\left(\dfrac{1}{3}\right)\sin^{-1}(u) + C = \left(\dfrac{1}{3}\right)\sin^{-1}\left(\dfrac{3x}{2}\right) + C$

The answer is (E).

PROBLEM 9. $\lim\limits_{x\to\infty} 4x\sin\left(\dfrac{1}{x}\right) =$

To solve this problem, you need to remember how to evaluate limits, particularly of trigonometric functions.

Step 1: As you should recall, the $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$. (This can be shown with L'Hopital's Rule. Differentiate the top and bottom of the limit to obtain $\lim\limits_{x\to 0}\dfrac{\cos x}{1}$, which just equals 1.) Thus, we need to find a way to convert this integral into one that looks like $\dfrac{\sin x}{x}$.

We can do this with a simple substitution. Let $y = \dfrac{1}{x}$. Now we can change this limit from $\lim\limits_{x\to\infty} 4x\sin\left(\dfrac{1}{x}\right)$ to $\lim\limits_{y\to 0} 4\left(\dfrac{\sin y}{y}\right)$

Step 2: $\lim\limits_{y\to 0} 4\left(\dfrac{\sin y}{y}\right) = 4(1) = 4$

The answer is (C).

PROBLEM 10. The position of a particle moving along the x-axis at time t is given by $x(t) = e^{\cos(2t)}, 0 \le t \le \pi$. For which of the following values of t will $x'(t) = 0$?

This problem requires you to know Derivatives of Exponential Functions, and Derivatives of Trigonometric Functions. Also, whenever you are given restrictions on the domain of a function, pay careful attention to the restrictions.

Step 1: $x'(t) = e^{\cos(2t)}\left(-2\sin(2t)\right)$

Step 2: Now that we have the derivative, set it equal to zero.

$$e^{\cos(2t)}\left(-2\sin(2t)\right) = 0$$

Step 3: Because $e^{\cos(2t)}$ can never equal zero (Did you know this? Make sure that you do!), we only have to set $-2\sin(2t)$ equal to zero. This will be true wherever $\sin(2t) = 0$. Because $\sin(t) = 0$ at $0, \pi, 2\pi, 3\pi,...$ we know thatAlthough there are an infinite number of solutions to this problem, because of the domain restriction we are only concerned with the first three solutions. Thus all three Roman Numeral answer choices work.

The answer is (E).

PROBLEM 11. $\lim\limits_{h \to 0} \dfrac{\sec(\pi + h) - \sec(\pi)}{h} =$

This may *appear* to be a limit problem, but it is *actually* testing to see whether you know The Definition of the Derivative.

Step 1: You should recall that $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = f'(x)$. Thus, if we replace $f(x)$ with $\sec(x)$, we can rewrite the problem as

$$\lim_{h \to 0} \frac{\sec(x+h) - \sec(x)}{h} = \left[\sec(x)\right]'$$

Step 2: The derivative of $\sec(x)$ is $\sec(x)\tan(x)$. Thus

$$\lim_{h \to 0} \frac{\sec(\pi + h) - \sec(\pi)}{h} = \sec(\pi)\tan(\pi)$$

Step 3: Because $\sec(\pi) = 1$ and $\tan(\pi) = 0$, this is equal to 0.

The answer is (B).

PROBLEM 12. Use differentials to approximate the change in the volume of a cube when the side is decreased from 8 to 7.99 cm. (in cm^3)

The volume of a cube is $V = x^3$. Using differentials, the change will be: $dV = 3x^2\,dx$

Substitute in $x = 8$ and $dx = -.01$, and we get:

$dV = 3(8^2)(-.01)$

$dV = -1.92$

The answer is (C).

PROBLEM 13. The radius of convergence of $\sum\limits_{n=1}^{\infty} \dfrac{a^n}{(x+2)^n}$; $a > 0$ is

Step 1:
An infinite series of the form $\sum\limits_{n=0}^{\infty} r^n$ will converge if $|r| < 1$. So all we have to do is to set $\left|\dfrac{a}{x+2}\right| < 1$.

Step 2: $\left|\dfrac{a}{x+2}\right| < 1$ means that $-1 < \dfrac{a}{x+2} < 1$. We can rewrite this as $-1 > \dfrac{x+2}{a} > 1$.

Step 3: Now we have $-a > x + 2 > a$ or $-a - 2 > x > a - 2$.

The answer is (C).

PROBLEM 14. $\int_0^1 \sin^{-1}(x)dx =$

Step 1: Your first reaction to this integral may very well be "I don't know how to find the integral of an inverse trigonometric function. I only know how to find the derivative of an inverse trigonometric function!" That's okay. This is actually an Integration By Parts problem. First of all, we are going to ignore the limits of integration until the end of this problem, and just focus on finding the integral itself. As you should recall, the formula for integration by parts is:

$$\int u\,dv = uv - \int v\,du$$

Step 2: Let

$$u = \sin^{-1} x \quad \text{and} \quad dv = dx$$

then

$$du = \frac{dx}{\sqrt{1-x^2}} \quad \text{and} \quad v = x$$

Now, using integration by parts, we have : $\int \sin^{-1}(x)dx = x\sin^{-1}(x) - \int \frac{xdx}{\sqrt{1-x^2}}$

Step 3: We can now solve this latter integral with u-substitution.

Let $u = 1 - x^2$ and $du = -2x\,dx$.

$$\frac{-1}{2}du = dx$$

Then we have: $\int \frac{dx}{\sqrt{1-x^2}} = \frac{-1}{2}\int u^{\frac{-1}{2}}du = -u^{\frac{1}{2}}$

Step 4: Substituting back for u gives us: $\int \sin^{-1}(x)dx = x\sin^{-1}(x) + \sqrt{(1-x^2)}$

Step 5: Now we evaluate at the limits of integration:

$$\left(x\sin^{-1}x + \sqrt{1-x^2} \right)\Big|_0^1 = \left((1)\sin^{-1}(1) + \sqrt{0}\right) - \left(0\sin^{-1}0 + \sqrt{1}\right) = \left(\frac{\pi}{2}\right) - (1) = \frac{\pi-2}{2}$$

The answer is (C).

PROBLEM 15. The equation of the line *normal* to $y = \sqrt{\dfrac{5-x^2}{5+x^2}}$ at $x = 2$ is

Step 1: This problem requires you to know how to find Equations of Tangent Lines. We will use the slope-intercept formula of a line: $(y - y_1) = m(x - x_1)$.

Step 2: When $x = 2$, $y = \sqrt{\dfrac{5 - (2^2)}{5 + (2^2)}} = \sqrt{\dfrac{5-4}{5+4}} = \sqrt{\dfrac{1}{9}} = \dfrac{1}{3}$, so

$$x_1 = 2 \quad \text{and} \quad y_1 = \dfrac{1}{3}$$

Step 3: $\dfrac{dy}{dx} = \dfrac{1}{2}\left(\dfrac{5-x^2}{5+x^2}\right)^{\frac{-1}{2}} \left(\dfrac{(5+x^2)(-2x) - (5-x^2)(2x)}{(5+x^2)^2}\right)$ This would now require some messy algebra to simplify, but fortunately we don't have to. We can Plug In 2 for x right now and solve for $\dfrac{dy}{dx} = \dfrac{1}{2}\left(\dfrac{5-2^2}{5+2^2}\right)^{\frac{-1}{2}} \left(\dfrac{(5+2^2)(-4) - (5-2^2)(4)}{(5+2^2)^2}\right)$,

which simplifies to $\dfrac{1}{2}\left(\dfrac{1}{9}\right)^{\frac{-1}{2}} \left(\dfrac{(9)(-4) - (1)(4)}{(9)^2}\right) = \dfrac{3}{2}\left(\dfrac{-36-4}{81}\right) = \dfrac{-20}{27}$.

Step 4: If we were finding the equation of a *tangent line*, we would use $\dfrac{-20}{27}$ for m in the equation, but, as you should recall, because we are finding the equation of the *normal line*, we use the *negative reciprocal* of $\dfrac{-20}{27}$ for m, which is $\dfrac{27}{20}$ and plug it into the equation of the line.

Step 5: Now we have $(y - y_1) = m(x - x_1)$ which becomes $y - \dfrac{1}{3} = \dfrac{27}{20}(x - 2)$.

Multiply through by 60 to get: $60y - 20 = 81x - 162$ or $81x - 60y = 142$.

The answer is (A).

PROBLEM 16. If c satisfies the conclusion of the Mean Value Theorem for Derivatives for $f(x) = 2\sin x$ on the interval $[0, \pi]$ then c is

Step 1: The Mean Value Theorem for Derivatives states that if a function is continuous on an interval $[a, b]$, then there exists some value c in that interval where

$$\dfrac{f(b) - f(a)}{b - a} = f'(c)$$

Step 2: $\dfrac{f(b)-f(a)}{b-a} = \dfrac{2\sin\pi - 2\sin 0}{\pi - 0} = \dfrac{0}{\pi} = 0$

Step 3: Thus $f'(c) = 0$. Because $f'(c) = 2\cos(c)$, we need to know what value of c makes $2\cos(c) = 0$. The value is $\dfrac{\pi}{2}$.

The answer is (C).

PROBLEM 17. The average value of $x \ln x$ on the interval $[1, e]$ is

This problem requires you to be familiar with the Mean Value Theorem for Integrals which we use to find the average value of a function.

Step 1: If you want to find the average value of $f(x)$ on an interval $[a, b]$, you need to evaluate the integral $\dfrac{1}{b-a}\int_a^b f(x)dx$. So here we would evaluate the integral

$$\frac{1}{e-1}\int_1^e x \ln x\, dx$$

Step 2: We are going to need to do integration by parts to evaluate this integral. Lets ignore the limits of integration for now and just do the integration.

$$\text{Let } u = \ln x \quad \text{and} \quad dv = xdx$$
$$\text{then}$$
$$du = \frac{1}{x}dx \quad \text{and} \quad v = \frac{x^2}{2}$$

Now the integral becomes: $\displaystyle\int x \ln x\, dx = \frac{x^2}{2}\ln x - \int\left(\frac{x^2}{2}\right)\left(\frac{1}{x}\right)dx = \frac{x^2}{2}\ln x - \frac{1}{2}\int x\, dx$

Thus we have: $\displaystyle\int_1^e x \ln x\, dx = \left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right)\Big|_1^e$

Step 3: $\left(\dfrac{x^2}{2}\ln x - \dfrac{x^2}{4}\right)\Big|_1^e = \left(\dfrac{e^2}{2}\ln e - \dfrac{e^2}{4}\right) - \left(\dfrac{1}{2}\ln 1 - \dfrac{1}{4}\right) = \left(\dfrac{e^2}{2} - \dfrac{e^2}{4} + \dfrac{1}{4}\right)$

This can be simplified to: $\dfrac{e^2 + 1}{4}$

Step 4: Don't forget to multiply by $\dfrac{1}{e-1}$!! This gives the final result of: $\dfrac{e^2+1}{4(e-1)}$

The answer is (D).

PROBLEM 18. A 17 foot ladder is sliding down a wall at a rate of –5 *feet/sec.* When the top of the ladder is 8 feet from the ground, how fast is the foot of the ladder sliding away from the wall (in *feet/sec.*)?

> **Step 1:** The ladder forms a right triangle with the wall, with the ladder itself as the hypotenuse. Whenever we see right triangles in related rates problems, we look to use the Pythagorean Theorem.
>
> Call the distance from the top of the ladder to the ground y, and the distance from the foot of the ladder to the wall x. Then the rate at which the top of the ladder is sliding down the wall is $\dfrac{dy}{dt}$, and the rate at which the foot of the ladder is sliding away from the wall is $\dfrac{dx}{dt}$, which is what we need to find. Now we use the Pythagorean Theorem to set up the relationships: $x^2 + y^2 = 17^2$
>
> **Step 2:** Differentiating both sides we obtain: $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$.
>
> **Step 3:** Because of the Pythagorean Theorem, we also know that, when $y = 8$, $x = 15$.
>
> **Step 4:** Now we plug everything into the equation from Step 2 and solve for $\dfrac{dx}{dt}$
>
> $$2(15)\dfrac{dx}{dt} + 2(8)(-5) = 0 \quad \text{and} \quad \dfrac{dx}{dt} = \dfrac{8}{3}$$

The answer is (B).

PROBLEM 19. If $\dfrac{dy}{dx} = 3y\cos x$, and $y = 8$ when $x = 0$, then $y =$

> **Step 1:** Separate the variables, by putting all of the terms containing y on the left hand side of the equals sign, and all of the terms containing x on the right hand side.
>
> $$\dfrac{dy}{y} = 3\cos x \, dx$$
>
> **Step 2:** Integrate both sides. $\displaystyle\int \dfrac{dy}{y} = 3\int \cos x \, dx$
>
> $$\ln y = 3\sin x + C$$

Whenever we have a differential equation where the solution is in terms of ln y, we always solve the equation for y. This involves raising e to the power of each side. This gives us: $e^{\ln y} = e^{3\sin x + C}$; $y = e^{3\sin x}e^C$; $y = Ce^{3\sin x}$. Now you should notice that e^C is just a constant, so we call that C (Confusing, isn't it?!), and write the equation as: $y = Ce^{3\sin x}$

Step 3: Now Plug In $y = 8$ and $x = 0$ in order to solve for C.

$8 = Ce^{3\sin 0} = Ce^0 = C$

$8 = C$

Step 4: This gives the final equation of $y = 8e^{3\sin x}$

The answer is (A).

PROBLEM 20. The length of the curve determined by $x = 3t$ and $y = 2t^2$ from $t = 0$ to $t = 9$ is

This problem requires you to find an Arc Length. This is a simple integral formula.

Step 1: The formula for the arc length of a curve given in parametric form on the interval $[a,b]$ is $\int_a^b \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\, dt$

Step 2: $\dfrac{dx}{dt} = 3$ and $\dfrac{dy}{dt} = 4t$, so $\int_a^b \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\, dt = \int_0^9 \sqrt{3^2 + (4t)^2}\, dt$

Step 3: $\int_0^9 \sqrt{3^2 + (4t)^2}\, dt = \int_0^9 \sqrt{9 + 16t^2}\, dt$

The answer is (E).

PROBLEM 21. If a particle moves in the xy-plane so that at time $t > 0$ its position vector is $\left(e^{t^2}, e^{-t^3}\right)$ then its velocity at time $t = 3$ is

This problem requires you to know how to find the velocity of a moving object.

Step 1: All you need to do to find the velocity of a moving object at a particular instant in time is to take the derivative of its position function at that time. The derivative of $e^{t^2} = 2te^{t^2}$ and the derivative of $e^{-t^3} = -3t^2e^{-t^3}$

Step 2: Now we Plug In 3 for t and we get $2te^{t^2} = 6e^9$ and $-3t^2e^{-t^3} = -27e^{-27}$

The answer is (D).

PROBLEM 22. The graph of $f(x) = \sqrt{11+x^2}$ has a point of inflection at

This problem requires you to know how to find the critical points on a graph, which is a crucial part of Graphing Functions.

Step 1: The points of inflection on a graph are generally at points where the second derivative is zero, but not necessarily at *all* points where the second derivative is zero. The good thing about the AP exam is that, in the multiple choice part of the test, you do not have to worry about exceptions to the second derivative rule. Thus, all we have to do here is to take the second derivative of the function and set it equal to zero.

$$f'(x) = \frac{1}{2}\left(11+x^2\right)^{\frac{-1}{2}}(2x) = \frac{x}{\left(11+x^2\right)^{\frac{1}{2}}}$$

and

$$f''(x) = \frac{\left(11+x^2\right)^{\frac{1}{2}}(1) - (x)\frac{1}{2}\left(11+x^2\right)^{\frac{-1}{2}}(2x)}{\left(11+x^2\right)}$$

$$= \frac{\left(11+x^2\right)^{\frac{1}{2}} - \frac{x^2}{\left(11+x^2\right)^{\frac{1}{2}}}}{\left(11+x^2\right)^{\frac{1}{2}}}$$

$$= \frac{\left(11+x^2\right) - x^2}{\left(11+x^2\right)^{\frac{3}{2}}}$$

$$= \frac{11}{\left(11+x^2\right)^{\frac{3}{2}}}$$

Step 2: When we are setting a rational function equal to zero, all we have to do is set the numerator equal to zero, and that value will be our point of inflection. Always double check that the value that makes the numerator equal to zero does not also make the DENOMINATOR equal to zero. If it does, this is NOT a point of inflection. Here, there is no point where 11 equals zero, so there is no point of inflection.

The answer is (E).

PROBLEM 23. What is the volume of the solid generated by rotating about the y-axis the region enclosed by $y = \sin x$ and the x-axis, from $x = 0$ to $x = \pi$?

This problem requires us to find the Volume of a Solid of Revolution.

Step 1: Whenever you find the volume of a solid of revolution, you should first draw the graph of the equation so that you are sure of exactly what the curve looks like. You can graph this easily on your calculator, and should get something that looks like this:
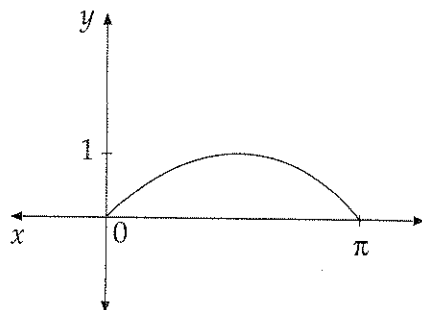
Step 2: Because we are revolving this curve around the y-axis, it will be easier to use the "shells" formula than the "washers" formula.

Using this formula, we get: $2\pi \int_0^\pi x \sin x \, dx$. This is one of our basic Integration By Parts integrals.

Let $u = x$ and $dv = \sin x \, dx$
        then
$du = dx$ and $v = -\cos x$

then $\int x \sin x \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x$

Step 3: Now we evaluate at the limits of integration and we get:

$$2\pi(-x \cos x + \sin x)\Big|_0^\pi = 2\pi\left[(-\pi \cos \pi + \sin \pi) - (0 + \sin 0)\right] = 2\pi^2$$

The answer is (B).

PROBLEM 24. $\displaystyle \int_{\frac{2}{\pi}}^{\infty} \frac{\sin\left(\frac{1}{t}\right)}{t^2} \, dt =$

This is an Improper Integral. As with many of the more difficult topics in Calculus, the AP examination only tends to ask us to solve very straightforward Improper Integrals. The trick is to change the improper integral into a proper one.

Step 1: First, we need to rewrite the integral as a limit. Let $a = \infty$, and evaluate:

$$\lim_{a \to \infty} \int_{\frac{2}{\pi}}^{a} \frac{\sin\left(\frac{1}{t}\right)}{t^2} dt.$$ This gets rid of the "infinity problem".

At this point, we ignore the limits of integration while we figure out how to do the integral.

Using u-substitution, let $u = \dfrac{1}{t}$ and $du = \dfrac{-1}{t^2} dt$

$$\int \frac{\sin\left(\frac{1}{t}\right)}{t^2} dt = -\int \sin u \, du = \cos u.$$ Substituting back, we get $\left. \cos\dfrac{1}{t} \right|_{\frac{2}{\pi}}^{a}$

Step 2: Now we evaluate the integral at the limits of integration and get:

$$\left. \cos\frac{1}{t} \right|_{\frac{2}{\pi}}^{a} = \cos\frac{1}{a} - \cos\frac{\pi}{2} = \cos\frac{1}{a}$$

Step 3: Finally, we take the limit and we get: $\displaystyle\lim_{a \to \infty} \cos\frac{1}{a} = \cos 0 = 1$

The answer is (A).

PROBLEM 25. A rectangle is to be inscribed between the parabola $y = 4 - x^2$ and the x-axis, with its base on the x-axis. The value of x that maximizes the area of the rectangle is

This is a Maximum/Minimum problem. It is usually helpful to draw a picture first, so we know what we are looking for.

Step 1: If we graph $y = 4 - x^2$ and sketch in a rectangle, we get something like this:
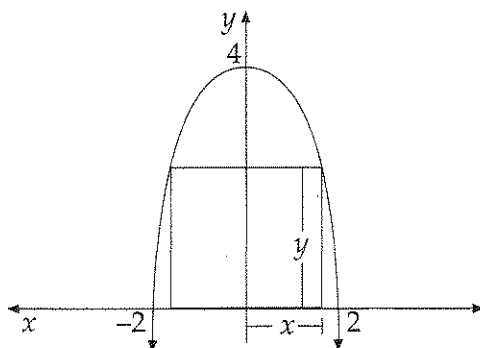
Step 2: Notice that the length of the base of the rectangle is $2x$ and the height of the rectangle is y. This means that the area of the rectangle is: $A = 2xy$. If we substitute $4 - x^2$ for y, we get: $A = 2x\left(4 - x^2\right) = 8x - 2x^3$.

Step 3: If we want to find the maximum area, all we have to do is take the derivative of A and set it equal to zero.

$$\frac{dA}{dx} = 8 - 6x^2 = 0$$

$$8 = 6x^2$$

$$x = \pm\sqrt{\frac{4}{3}} = \pm\frac{2}{\sqrt{3}}$$

Because the answer is looking for a length, we are satisfied with the positive answer. Besides, $-\dfrac{2}{\sqrt{3}}$ isn't an answer choice.

The answer is (B).

PROBLEM 26. $\displaystyle\int \frac{dx}{\sqrt{9 - x^2}} =$

If you look at this integral, you should notice that it is similar to the integral

$$\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x + C$$

Step 1: If we factor 9 out of the radicand, we can convert our integral into the one above.

$$\int \frac{dx}{\sqrt{9 - x^2}} = \int \frac{dx}{\sqrt{9\left(1 - \dfrac{x^2}{9}\right)}} = \int \frac{dx}{3\sqrt{\left(1 - \dfrac{x^2}{9}\right)}} = \frac{1}{3}\int \frac{dx}{\sqrt{\left(1 - \dfrac{x^2}{9}\right)}}$$

Step 2: Now we can use u-substitution. Let $u = \dfrac{x}{3}$. Then $du = \dfrac{1}{3}dx$ and $3du = dx$. We get:

$$\frac{1}{3}\int\frac{dx}{\sqrt{\left(1-\dfrac{x^2}{9}\right)}}=\frac{1}{3}\int\frac{3du}{\sqrt{1-u^2}}=\int\frac{du}{\sqrt{1-u^2}}=\sin^{-1}u+C$$

Step 3: Substituting back, we get:

$$\sin^{-1}\frac{x}{3}+C$$

The answer is (D).

PROBLEM 27. Find $\displaystyle\lim_{x\to\infty}x^{\frac{1}{x}}$

This is a limit of an *indeterminate form*. We use L'Hopital's rule to evaluate these limits, but we need to get this into the form $\dfrac{f(x)}{g(x)}$ so that we can take the derivative of the top and bottom. We do this using logarithms.

Step 1: First, let $y=x^{\frac{1}{x}}$

Step 2: Now, take the log of both sides (remember, when we say log, we mean *natural log*, not *common log*).

We get: $\ln y=\ln x^{\frac{1}{x}}$, which we can rewrite, using log rules, as $\dfrac{1}{x}\ln x$ or $\dfrac{\ln x}{x}$

Step 3: Now we can use L'Hopital's rule. Take the derivative of the top and bottom.

$$\lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{\dfrac{1}{x}}{1}=0$$

Be careful! This is NOT the answer! We just found that that the limit of $\ln y\to 0$, therefore, $y\to e^0=1$. Whenever you use this technique, remember to reverse the logarithm at the end.

The answer is (B).

PROBLEM 28. What is the sum of the Maclaurin series $\pi - \dfrac{\pi^3}{3!} + \dfrac{\pi^5}{5!} - \dfrac{\pi^7}{7!} + \ldots$

$+(-1)^n \dfrac{\pi^{2n+1}}{(2n+1)!} + \ldots$?

This series is of the form $x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \ldots + (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} + \ldots$. This, as we know, is the Maclaurin Series expansion of $\sin x$. In the problem, we simply have π instead of x. Therefore, this is equal to $\sin \pi = 0$.

The answer is (B).

PROBLEM 29. The first three non-zero terms in the Taylor series about $x = 0$ of $\cos x$ are

This is a Taylor Series problem. There are four Taylor Series that you should memorize. *This is one of them.* But just in case you didn't memorize it . . .

Step 1: The formula for a Taylor Series about the point $x = a$ is:

$$f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \ldots + f^{(n)}(a)\frac{(x-a)^n}{n!} + \ldots$$

Step 2: First, let's take the first few derivatives of $\cos x$:

$$f(x) = \cos x$$
$$f'(x) = -\sin x$$
$$f''(x) = -\cos x$$
$$f'''(x) = \sin x$$
$$f^{(4)}(x) = \cos x$$

Step 3: Next, we evaluate each of these at $a = 0$:

$$f(0) = \cos 0 = 1$$
$$f'(0) = -\sin 0 = 0$$
$$f''(0) = -\cos 0 = -1$$
$$f'''(0) = \sin 0 = 0$$
$$f^{(4)}(0) = \cos 0 =$$

Step 4: Now if we Plug In 0 for a throughout the formula we get:

$$f(0) + f'(0)(x) + f''(0)\frac{(x)^2}{2!} + f'''(0)\frac{(x)^3}{3!} + f^{(4)}(0)\frac{(x)^4}{4!} + \ldots$$

$$= 1 + (0)(x) + (-1)\left(\frac{x^2}{2!}\right) + (0)\left(\frac{x^3}{3!}\right) + (1)\left(\frac{x^4}{4!}\right) + \ldots$$

$$= 1 + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

The answer is (C).

You should make sure to memorize the four Taylor Series in the unit. One of them almost always shows up on the AP Examination!

PROBLEM 30. $\int \cos^3 x\, dx =$

This is a Trigonometric Integral. Generally, these are solved by using the trigonometric substitutions that you learned in precalculus. If you are unfamiliar with these, you should go back and review them.

Step 1: First, rewrite $\cos^3 x$ as $\cos x(\cos^2 x)$. Then, because $\cos^2 x = (1 - \sin^2 x)$, we can rewrite the integral as

$$\int \cos^3 x\, dx = \int \cos x(1 - \sin^2 x)dx = \int \cos x\, dx - \int \cos x \sin^2 x\, dx$$

Step 2: The first integral is easy. $\int \cos x\, dx = \sin x$. We do the second integral with u-substitution.

Let $u = \sin x$ and $du = \cos x dx$. Then $\int \cos x \sin^2 x\, dx = \int u^2 du = \frac{u^3}{3} + C$

Substituting back for u and combining, gives us

$$\int \cos x\, dx - \int \cos x \sin^2 x\, dx = \sin x - \frac{\sin^3 x}{3} + C$$

The answer is (C).

PROBLEM 31. If $f(x) = (3x)^{(3x)}$ then $f'(x) =$

This problem requires us to find the Derivative of an Exponential Function. Any power of x can be written as a power of e, which is what we will use to do this derivative.

Step 1: Rewrite $(3x)^{(3x)}$ as $e^{(3x)\ln(3x)}$. Now, we can take the derivative of this using the Chain Rule and the Product Rule.

Step 2: $f'(x) = \left[(3x)\left(\frac{3}{3x}\right) + 3\ln(3x)\right]e^{(3x)\ln(3x)} = (3 + 3\ln(3x))e^{(3x)\ln(3x)}$

Step 3: Now, replacing $e^{(3x)\ln(3x)}$ with $(3x)^{(3x)}$, gives us $(3x)^{(3x)}(3+3\ln(3x))$.

The answer is (A).

PROBLEM 32. To what limit does the sequence $s_n = \dfrac{3+n}{3^n}$ converge as n approaches infinity?

This will require us to use one of the convergence tests for Infinite Series. If we use the Ratio Test, we will be able to tell if the sequence converges, and, if so, to what value.

Step 1: $s_n = \dfrac{3+n}{3^n}$ and $s_{n+1} = \dfrac{4+n}{3^{n+1}}$. So the Ratio Test says that, if $\lim\limits_{n\to\infty}\left|\dfrac{s_{n+1}}{s_n}\right| < 1$, then

the sequence will converge to zero, and if $\lim\limits_{n\to\infty}\left|\dfrac{s_{n+1}}{s_n}\right| > 1$, then the series diverges–in

other words, it has no limit.

Step 2: $\dfrac{s_{n+1}}{s_n} = \dfrac{\dfrac{4+n}{3^{n+1}}}{\dfrac{3+n}{3^n}} = \dfrac{4+n}{3+n}\cdot\dfrac{3^n}{3^{n+1}} = \dfrac{1}{3}\left(\dfrac{4+n}{3+n}\right)$

If we take the limit, $\lim\limits_{n\to\infty}\dfrac{1}{3}\left(\dfrac{4+n}{3+n}\right) = \dfrac{1}{3}$.

Because this is less than one, the sequence converges to zero.

The answer is (C).

PROBLEM 33. $\displaystyle\int \dfrac{18x-17}{(2x-3)(x+1)}\,dx =$

This is another Partial Fractions integral.

Step 1: Write the integrand as: $\dfrac{18x-17}{(2x-3)(x+1)} = \dfrac{A}{(2x-3)} + \dfrac{B}{(x+1)}$

Step 2: Multiply both sides by: $(2x-3)(x+1)$

Step 3: Now we have: $18x - 17 = A(x+1) + B(2x-3)$

Step 4: $18x - 17 = Ax + A + 2Bx - 3B$

Step 5: $18x - 17 = (Ax + 2Bx) + (A - 3B)$

Step 6: $18x - 17 = x(A + 2B) + (A - 3B)$

Step 7: Thus: $(A + 2B) = 18$ and $(A - 3B) = -17$

Step 8: Solve this using simultaneous equations to get: $A = 4$ and $B = 7$

Step 9: We can now rewrite the original integral as:

$$\int\left(\frac{4}{2x-3}+\frac{7}{x+1}\right)dx = 4\int\frac{dx}{2x-3}+7\int\frac{dx}{x+1}$$

Step 10: These are both basic ln integrals, and we get:

$$4\int\frac{dx}{2x-3}+7\int\frac{dx}{x+1} = 2\ln|2x-3|+7\ln|x+1|+C$$

The answer is (B).

PROBLEM 34. A particle moves along a path described by $x = \cos^3 t$ and $y = \sin^3 t$. Find the distance that the particle travels along the path from $t = 0$ to $t = \frac{\pi}{2}$.

This is another Arc Length problem.

Step 1: The formula for the arc length of a curve given in parametric form on the interval $[a,b]$ is $\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

Step 2: $\dfrac{dx}{dt} = -3\cos^2 t\sin t$ and $\dfrac{dy}{dt} = 3\sin^2 t\cos t$, so

$$\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt = \int_0^{\frac{\pi}{2}}\sqrt{\left(-3\cos^2 t\sin t\right)^2+\left(3\sin^2 t\cos t\right)^2}\,dt$$

$$= \int_0^{\frac{\pi}{2}}\sqrt{\left(9\cos^4 t\sin^2 t\right)+\left(9\sin^4 t\cos^2 t\right)}\,dt$$

Step 3: Now you have a choice. Integrate this using your calculator. You should get 1.5.

If you are not comfortable with this on the calculator, or if you prefer to do the integration, you need to do the following. Reduce the integrand by factoring out $\sin^2 t\cos^2 t$ and we get:

$$\int_0^{\frac{\pi}{2}}\sqrt{9\left(\sin^2 t\cos^2 t\right)\left(\sin^2 t+\cos^2 t\right)}\,dt = \int_0^{\frac{\pi}{2}}\sqrt{9\left(\sin^2 t\cos^2 t\right)(1)}\,dt$$

$$\int_0^{\frac{\pi}{2}}\sqrt{9\sin^2 t\cos^2 t}\,dt = \int_0^{\frac{\pi}{2}}3\sin t\cos t\,dt$$

Step 4: Using u-substitution, let $u = \sin t$ and $du = \cos t$. Then we get:

$$\int_0^{\frac{\pi}{2}} 3\sin t \cos t \, dt = 3\int_0^1 u\,du = 3\left(\frac{u^2}{2}\right)\bigg|_0^1 = \frac{3}{2}$$

The answer is (B).

PROBLEM 35. The sales price of an item is $800 - 35x$ dollars and the total manufacturing cost is $2x^3 - 140x^2 + 2600x + 10000$ dollars, where x is the number of items. What number of items should be manufactured in order to optimize the manufacturer's total profit?

Anytime that you see the word "optimize", you will be doing a Maximum/Minimum problem. The profit is the number of items sold times the difference between the sales price of each object and its cost. The number of items is x, so the total sales price is $800x - 35x^2$.

Step 1: Let P equal profit.

$P = 800x - 35x^2 - (2x^3 - 140x^2 + 2600x + 10000) = -2x^3 + 105x^2 - 1800x - 10000$

Step 2: $\dfrac{dP}{dx} = -6x^2 + 210x - 1800$. Setting it equal to 0 we get:

$$-6x^2 + 210x - 1800 = 0$$

$$x^2 - 35x + 300 = 0$$

$$(x - 20)(x - 15) = 0$$

$$x = 15, 20$$

Step 3: In order to determine which of these is the maximum and which is the minimum, use the second derivative test. The second derivative of profit is

$\dfrac{d^2P}{dx^2} = -12x + 210$. At $x = 15$, we get $\dfrac{d^2P}{dx^2} = 30$, so this is a minimum. At $x = 20$, we get $\dfrac{d^2P}{dx^2} = -30$, so this is a maximum. Therefore, the optimum number of units is 20.

The answer is (E).

PROBLEM 36. Find the area enclosed by the polar equation $r = 4 + \cos\theta$ for $0 \le \theta \le 2\pi$.

This problem requires you to know the polar formula for finding the area of a region. The formula is $Area = \dfrac{1}{2}\int_{\theta=a}^{\theta=b} r^2 d\theta$.

Step 1: $\dfrac{1}{2}\int_{\theta=a}^{\theta=b} r^2 d\theta = \dfrac{1}{2}\int_0^{2\pi}(4 + \cos\theta)^2 d\theta$. Use your calculator to get 51.8363.

The answer is (D).

If you don't want to do this integral with the calculator, do the following:

If we expand the integrand, we will get three integrals:

$$\frac{1}{2}\int_0^{2\pi}(4+\cos\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi}16\,d\theta + \frac{1}{2}\int_0^{2\pi}8\cos\theta\,d\theta + \frac{1}{2}\int_0^{2\pi}\cos^2\theta\,d\theta$$

The first two integrals are easy and give us:

$$\frac{1}{2}(16\theta)\Big|_0^{2\pi} + \frac{1}{2}(8\sin\theta)\Big|_0^{2\pi} + \frac{1}{2}\int_0^{2\pi}\cos^2\theta\,d\theta$$

Evaluating the first two integrals, we get $(8\theta)\Big|_0^{2\pi} + (4\sin\theta)\Big|_0^{2\pi} = 16\pi$.

Step 2: We do the second integral using a trigonometric substitution.

$$\frac{1}{2}\int_0^{2\pi}\cos^2\theta\,d\theta = \frac{1}{2}\int_0^{2\pi}\frac{1+\cos 2\theta}{2}\,d\theta$$

If we simplify this integral, we get

$$\frac{1}{2}\int_0^{2\pi}\frac{1}{2}\,d\theta + \frac{1}{2}\int_0^{2\pi}\frac{\cos 2\theta}{2} = \frac{1}{4}\theta\Big|_0^{2\pi} + \frac{1}{8}\sin 2\theta\Big|_0^{2\pi} = \frac{\pi}{2}$$

If we add $16\pi + \dfrac{\pi}{2}$, we get $\dfrac{33\pi}{2}$.

The answer is (D).

PROBLEM 37. Use the trapezoid rule with $n = 4$ to approximate the area between the curve $f(x) = x^3 - x^2$ and the x-axis from $x = 3$ to $x = 4$.

This problem will require you to be familiar with the Trapezoid Rule. This is very easy to do on the calculator, and some of you may even have written programs to evaluate this. Even if you haven't, the formula is easy. The area under a curve from $x = a$ to $x = b$, divided into n intervals is approximated by the Trapezoid Rule and is

$$\left(\frac{1}{2}\right)\left(\frac{b-a}{n}\right)\left[y_0 + 2y_1 + 2y_2 + 2y_3 \ldots + 2y_{n-2} + 2y_{n-1} + y_n\right]$$

This formula may look scary, but it actually is quite simple, and the AP Examination never uses a very large value for n anyway.

Step 1: $\dfrac{b-a}{n} = \dfrac{4-3}{4} = \dfrac{1}{4}$ Plugging into the formula, we get:

$$\frac{1}{8}\left[\left(3^3 - 3^2\right) + 2\left(3.25^3 - 3.25^2\right) + 2\left(3.5^3 - 3.5^2\right) + 2\left(3.75^3 - 3.75^2\right) + \left(4^3 - 4^2\right)\right]$$

This is easy to Plug Into your calculator and you will get 31.516.

The answer is (D).

PROBLEM 38. If $f(x) = \displaystyle\sum_{k=0}^{\infty}\left(\cos^2 x\right)^k$, then $f\left(\dfrac{\pi}{4}\right)$ is

Step 1: If a series is of the form $\displaystyle\sum_{n=0}^{\infty} ar^n$, and $|r| < 1$ then the sum of the series is $(a)\left(\dfrac{1}{1-r}\right)$, where a is the first term. (Notice that if the series went from one to infinity, instead of zero to infinity, then the formula would be $(a)\left(\dfrac{r}{1-r}\right)$. Be careful that you memorize the correct formula. You could avoid this confusion by deriving the formula as you do the problem. To learn how to do that, refer to the unit on Infinite Series.) Here r is $\cos^2 x$ and a is $\dfrac{1}{2}$, and $\left|\cos^2 x\right| < 1$. This is very important.

If $|r| \geq 1$ the formula doesn't work. So the sum is $\left(\dfrac{1}{2}\right)\left(\dfrac{1}{1 - \cos^2 x}\right)$.

Using trigonometric substitution, this sum can be simplified to $\left(\dfrac{1}{2}\right)\dfrac{1}{\sin^2 x}$

Now if we Plug In $\dfrac{\pi}{4}$ for x, we get $\left(\dfrac{1}{2}\right)\dfrac{1}{\sin^2\left(\dfrac{\pi}{4}\right)} = 1$

The answer is D.

PROBLEM 39. The volume of the solid that results when the area between the graph of $y = x^2 + 2$ and the graph of $y = 10 - x^2$ from $x = 0$ to $x = 2$ is rotated around the x-axis is:

This is another Volume of a Solid of Revolution problem. As you should have noticed by now, these are very popular on the AP Examination and show up in both the multiple choice section and in the Long Problem section. If you are not good at these, go back and review the unit carefully. You cannot afford to get these wrong on the AP! The good thing about this volume problem is that it is in the calculator part of the multiple choice section, so you can use a program and your graphing calculator to assist you with this problem.

Step 1: First, graph the two curves on the same set of axes. The graph should look like this:
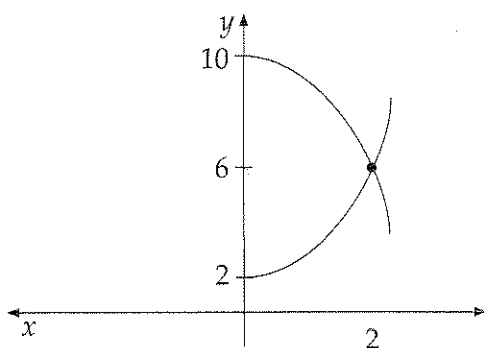
Step 2: Now look at the answer choices. Notice that each answer choice is the sum of two integrals, and all of the functions are in terms of y. This means that you are going to have to use the method of shells, and that you will have to convert the functions from being in terms of x to being in terms of y. Also, when you make a horizontal slice in this region, you are using different curves above and below the intersection point, so you will have to do two integrals. Okay. One thing at a time.

Step 3: First let's find where the two curves intersect. We can do this either with the calculator or algebraically.

Algebraically, we set the two equations equal to each other.

$$x^2 + 2 = 10 - x^2$$

$$2x^2 = 8$$
$$x^2 = 4$$
$$x = \pm 2$$

We will use $x = 2$, and Plugging In 2 for x, we get $y = 6$.

Now we have to find what y is when $x = 0$. On the lower curve, $y = 2$. On the higher curve, $y = 10$. These then are our limits of integration. We will have two integrals, one from 2 to 6, the other from 6 to 10. By process of elimination, this means that the answer can only be B or C.

Step 4: Now we have to convert each of the equations to being in terms of y.

$$y = x^2 + 2 \qquad y = 10 - x^2$$
$$x^2 = y - 2 \qquad x^2 = 10 - y$$
$$x = \pm\sqrt{y - 2} \qquad x = \pm\sqrt{10 - y}$$

Step 5: We are only concerned with the positive roots for this region, so using the Shells formula, we get that the volume is $2\pi\int_2^6 y\left(\sqrt{y-2}\right)dy + 2\pi\int_6^{10} y\left(\sqrt{10-y}\right)dy$.

The answer is (B).

PROBLEM 40. $\displaystyle\int_0^4 \frac{dx}{\sqrt{9 + x^2}} =$

Remember, this is the calculator part of the test. You can simply evaluate this on your calculator using **fnint**. If you are not comfortable with this, we will show you how to do this algebraically. This integral requires a Trigonometric Substitution. These types of integrals require a lot of algebra, so you should leave them for the second pass.

Step 1: Using your calculator, enter **fnint** $((1/(\sqrt{(9 + x^2)}))$, x, 0, 4). You should get 1.0986. Evaluate each of the answer choices to see which has the same value, or which is closest.

The answer is (A).

Step 2: If you are not using your calculator, you first have to do a substitution. Let's ignore the limits of integration for now, and just evaluate the integral. Whenever we have an integral of the form $\sqrt{(a^2 + x^2)}$, we do the substitution $x = a\tan\theta$. So here we let $x = 3\tan\theta$ and $dx = 3\sec^2\theta\,d\theta$

Substituting into the integrand, we get: $\displaystyle\int \frac{dx}{\sqrt{9 + x^2}} = \int \frac{3\sec^2\theta}{\sqrt{9 + 9\tan^2\theta}}\,d\theta$

If we factor 9 out of the radical in the denominator, we get :

$$\int \frac{3\sec^2\theta}{\sqrt{9+9\tan^2\theta}}\,d\theta = \int \frac{3\sec^2\theta}{3\sqrt{1+\tan^2\theta}}\,d\theta = \int \frac{\sec^2\theta}{\sec\theta}\,d\theta = \int \sec\theta\,d\theta$$

You should have memorized this integral. It is $\ln|\sec\theta + \tan\theta|$.

Step 3: Now we want to evaluate the limits of integration, but in order to do that, we should substitute back for x. If $x = 3\tan\theta$ then $\frac{x}{3} = \tan\theta$. Using the Pythagorean Theorem, $\frac{\sqrt{9+x^2}}{3} = \sec\theta$

So now we have $\ln|\sec\theta + \tan\theta| = \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right|$. If we evaluate the limits of integration, we get:

$$\ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right|\Bigg|_0^4 = \ln\left|\frac{5}{3} + \frac{4}{3}\right| - \ln|1| = \ln 3$$

The answer is (A).

PROBLEM 41. The rate that an object cools is directly proportional to the difference between its temperature (in Kelvin) at that time and the surrounding temperature (in Kelvin). If an object is initially at $35K$, and the surrounding temperature remains constant at $10K$, it takes 5 minutes for the object to cool to $25K$. How long will it take for the object to cool to $20K$?

This is a Differential Equation. Each AP Examination tends to contain one differential equation word problem. They usually give you the same type of equation and are actually not terribly difficult, once you understand the question.

Step 1: The first sentence tells us what the equation is going to be. Let T stand for temperature at a particular time, and S stand for the surrounding temperature. Then our equation is: $\frac{dT}{dt} = k(T-S)$. Time is always represented by t, and k is a constant.

This equation is solvable using separation of variables. Put everything that contains a T on the left side, and everything that contains a t on the right side:

$$\frac{dT}{(T-S)} = kdt$$

If we integrate both sides we get: $\int \frac{dT}{(T-S)} = k \int dt$

and performing the integration gives us: $\ln|T-S| = kt + C$.

Step 2: Whenever we have an equation of this form, we then exponentiate both sides, giving us:

$|T-S| = e^{kt+C}$ or $|T-S| = Ce^{kt}$

If we Plug In the rest of the information from the problem, we can solve for the constants k and C.

The initial temperature tells us that at time $t = 0$, $T = 35$, and $S = 10$. So $|35-10| = Ce^{k(0)}$ and $25 = C$

Then at time $t = 5$, $T = 25$. So, $|25-10| = 25e^{k(5)}$

$$15 = 25e^{k(5)}$$
$$\frac{3}{5} = e^{5k}$$
$$\frac{1}{5}\ln\left(\frac{3}{5}\right) = k$$

This gives us the final equation $|T-10| = 25e^{\left(\frac{1}{5}\ln\left(\frac{3}{5}\right)\right)(t)}$

Step 3: Finally, we Plug In the last bit of information, that $T = 20$ to solve for t.

$$|20 - 10| = 25e^{\left(\frac{1}{5}\ln\left(\frac{3}{5}\right)\right)(t)}$$

$$\frac{2}{5} = e^{\left(\frac{1}{5}\ln\left(\frac{3}{5}\right)\right)(t)}$$

$$\ln\frac{2}{5} = \frac{1}{5}\ln\left(\frac{3}{5}\right)(t)$$

$$5\frac{\ln(2/5)}{\ln(3/5)} = t$$

$$t = 8.97$$

The answer is (D).

PROBLEM 42. $\displaystyle\int e^x \cos x \, dx =$

You should recognize this integral as an elementary **Integration By Parts** integral. If so, this won't be very hard to do.

Step 1: Let $u = e^x$ and $dv = \cos x \, dx$

$\qquad du = e^x dx \qquad\qquad v = \sin x$

Then we have: $\displaystyle\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$

Step 2: We need to do integration by parts *a second time* to evaluate the second integral.

$$\text{Let } u = e^x \quad \text{and} \quad dv = \sin x \, dx$$
$$du = e^x dx \qquad\qquad v = -\cos x$$

Now we have: $\displaystyle\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$

Step 3: Although this looks as if we are back where we started, and will have to do a *third* integration, if we add $\displaystyle\int e^x \cos x \, dx$ to both sides we get:

$$2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x$$

Now if we divide both sides by 2 we get:

$$\int e^x \cos x \, dx = \frac{e^x \sin x + e^x \cos x}{2}$$
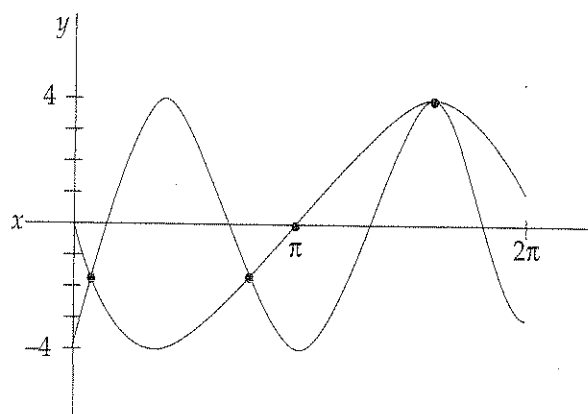
The answer is (A).

PROBLEM 43. Two particles leave the origin at the same time and move along the y-axis with their respective positions determined by the functions $y_1 = \cos 2t$ and $y_2 = 4 \sin t$ for $0 < t < 6$. For how many values of t do the particles have the same acceleration?

If you want to find acceleration, all you have to do is take the second derivative of the position functions.

Step 1: $\dfrac{dy_1}{dx} = -2\sin 2t$ and $\dfrac{dy_2}{dx} = 4\cos t$

$\dfrac{d^2 y_1}{dx^2} = -4\cos 2t$ and $\dfrac{d^2 y_2}{dx^2} = -4\sin t$

Step 2: Now all we have to do is to graph both of these equations on the same set of axes on a calculator. You should make the window from $x = 0$ to $x = 7$ (leave yourself a little room so that you can see the whole range that you need). You should get a picture that looks like this:

Where the graphs intersect, the acceleration is the same. There are three points of intersection.

The answer is (D).

PROBLEM 44: The minimum value of the function $y = x^3 - 7x + 11$; $x \geq 0$ is approximately

You have two options. First, let's do the problem without using the calculator.

If we want to find the minimum, we take the derivative and set it equal to zero. We get: $\frac{dy}{dx} = 3x^2 - 7 = 0$. Now we solve for x:

$$3x^2 = 7$$

$$x^2 = \frac{7}{3}$$

$$x = \pm\sqrt{\frac{7}{3}}$$

We are only concerned with positive values of x (note the restriction in the problem), so let's look at $x = \sqrt{\frac{7}{3}}$. First, we have to determine whether this is a minimum or a maximum. The simplest way to do this is with the second derivative test. Take the second derivative and we get: $\frac{d^2y}{dx^2} = 6x$. Now, Plug In $x = \sqrt{\frac{7}{3}}$. The second derivative is positive there, so the point is a minimum.

Now, we plug $x = \sqrt{\frac{7}{3}}$ back into the original equation to find the y value.

$$y = \left(\sqrt{\frac{7}{3}}\right)^3 - 7\left(\sqrt{\frac{7}{3}}\right) + 11 \approx 3.872 \text{ (rounded to 3 decimal places)}.$$

Now let's do this using the calculator.

Using the TI 82/83, press Y= and enter the graph as $Y_1 = x^\wedge 3 - 7x + 11$.

The answer is (E).

PROBLEM 45: Use Euler's method with $h = 0.2$ to estimate $y(1)$, if $y' = y$ and $y(0) = 1$.

We are given that the curve goes through the point $(0,1)$. We will call the coordinates of this point $x_0 = 0$ and $y_0 = 1$. The slope is found by plugging $y_0 = 1$ into $y' = y$, so we have an initial slope of $y_0' = 1$.

Now we need to find the next set of points.

Step 1: Increase x_0 by h to get x_1.

$$x_1 = 0.2$$

Step 2: Multiply h by y_0' and add to y_0 to get y_1.

$$y_1 = 1 + 0.2(1) = 1.2$$

Step 3: Find y_1' by plugging y_1 into the equation for y'

$$y_1' = 1.2$$

Repeat until you get to $x = 1$.

Step 1: Increase x_1 by h to get x_2.

$$x_2 = 0.4$$

Step 2: Multiply h by y_1' and add to y_1 to get y_2.

$$y_2 = 1.2 + 0.2(1.2) = 1.44$$

Step 3: Find y_2' by plugging y_2 into the equation for y'

$$y_2' = 1.44$$

Step 1: $x_3 = x_2 + h$

$$x_3 = 0.6$$

Step 2: $y_3 = y_2 + h(y_2')$

$$y_3 = 1.44 + 0.2(1.44) = 1.728$$

Step 3: $y_3' = y_3$

$$y_3' = 1.728$$

Step 1: $x_4 = x_3 + h$

$$x_4 = 0.8$$

Step 2: $y_4 = y_3 + h(y_3')$

$$y_4 = 1.728 + 0.2(1.728) = 2.0736$$

Step 3: $y_4' = y_4$

$$y_4' = 2.0736$$

Step 1: $x_5 = x_4 + h$.

$$x_5 = 1.0$$

Step 2: $y_5 = y_4 + h(y_4')$

$$y_5 = 2.0736 + 0.2(2.0736) = 2.48832$$

We don't need to go any farther because we are asked for the value of y when $x = 1$.

The answer is $y = 2.48832$.

The answer is (C).

ANSWERS AND EXPLANATIONS TO SECTION II

PROBLEM 1. Two particles travel in the xy-plane. For time $t \geq 0$, the position of particle A is given by $x = t + 1$ and $y = (t + 1)^2 - 2t - 2$, and the position of particle B is given by $x = 4t - 2$ and $y = -2t + 2$.

(a) Find the velocity vector for each particle at time $t = 2$.

Step 1: Because velocity is the derivative of position with respect to time, all we have to do is take the derivative of each of the position functions.

Particle A: The position of the x-coordinate is $t + 1$, the velocity is 1.

The position of the y-coordinate is $(t + 1)^2 - 2t - 1$, the velocity is $2(t + 1)(1) - 2 = 2t$

Particle B: The position of the x-coordinate is $4t - 2$, the velocity is 4.

The position of the y-coordinate is $-2t + 2$, the velocity is -2.

Step 2: Now all we have to do is Plug In.

At $t = 2$, particle A's velocity vector is $(1, 4)$.

At $t = 2$, particle B's velocity vector is $(4, -2)$.

(b) Set up an integral expression for the distance traveled by particle A from time $t = 1$ to $t = 3$. Do not evaluate the integral.

Step 1: We are being asked to find the arc length of a curve in parametric form. The formula is: $\int_a^b \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\, dt$, where the starting time is a, and the ending time is b.

Plugging into the formula, we get: $\displaystyle\int_1^3 \sqrt{(1)^2 + (2t)^2}\, dt = \int_1^3 \sqrt{1 + 4t^2}\, dt$

(c) At what time do the two particles collide? Justify your answer.

Step 1: The particles collide when they have the same x and y-coordinates. If we set the equations for the x-coordinates of the two particles equal we get:

$$t + 1 = 4t - 2 \text{ so } t = 1.$$

At this time, the y-coordinate for particle A is: $(1+1)^2 - 2(1) - 2 = 0$.

The y-coordinate for particle B is: $-2(1) + 2 = 0$. These are the same, so at time $t = 1$, the two particles collide.

(d) Sketch the path of both particles from time $t = 0$ to $t = 4$. Indicate the direction of each particle along its path.

By shifting the calculator into parametric mode, setting the window to match the one below, and inserting the equations for particles A and B into $x_1, y_1, x_2,$ and y_2 respectively, we get:
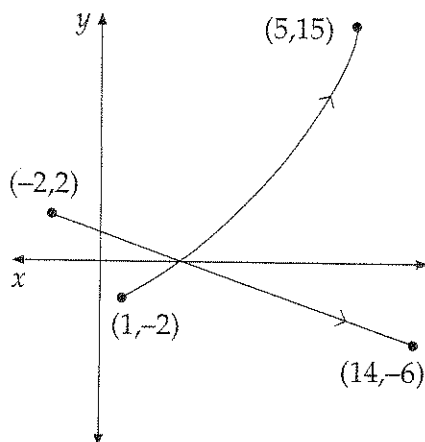
PROBLEM 2. Let f be the function given by $y = f(x) = 2x^4 - 4x^2 + 1$.

(a) Find an equation of the line tangent to the graph at $(-2, 17)$.

In order to find the Equation of a Tangent Line at a particular point we need to take the derivative of the function and Plug In the x and y values at that point to give us the slope of the line.

Step 1: The derivative is: $f'(x) = 8x^3 - 8x$. If we Plug In $x = -2$, we get:

$f'(-2) = 8(-2)^3 - 8(-2) = -48$. This is the slope m.

Step 2: Now we use the slope-intercept form of the equation of a line, $y - y_1 = m(x - x_1)$, and Plug In the appropriate values of x, y, and m.

$y - 17 = -48(x + 2)$. If we simplify this we get $y = -48x - 79$.

(b) Find the x and y-coordinates of the relative maxima and relative minima.

If we want to find the maxima/minima, we need to take the derivative and set it equal to zero. The values that we get are called critical points. We will then test each point to see if it is a maximum or a minimum.

Step 1: We already have the first derivative from part (a), so we can just set it equal to zero: $8x^3 - 8x = 0$.

If we now solve this for x we get:

$$8x(x^2 - 1) = 0 \quad 8x(x + 1)(x - 1) = 0 \quad x = 0, 1, -1$$

These are our critical points. In order to test if a point is a maximum or a minimum, we usually use the *second derivative test*. We plug each of the critical points into the second derivative. If we get a positive value, the point is a relative minimum. If we get a negative value, the point is a relative maximum. If we get zero, the point is a point of inflection.

Step 2: The second derivative is $f''(x) = 24x^2 - 8$. If we Plug In the critical points we get:

$$f''(0) = 24(0)^2 - 8 = -8$$
$$f''(1) = 24(1)^2 - 8 = 16$$
$$f''(-1) = 24(-1)^2 - 8 = 16$$

So $x = 0$ is a relative maximum, and $x = 1, -1$ are relative minima.

Step 3: In order to find the y-coordinates, we plug the x values back into the original equation, and solve.

$$f(0) = 1$$
$$f(1) = -1$$
$$f(-1) = -1$$

and our points are

$(0,1)$ is a relative maximum

$(1,-1)$ is a relative minimum

$(-1,-1)$ is a relative minimum

(c) Find the x-coordinates of the points of inflection.

If we want to find the points of inflection, we set the second derivative equal to zero. The values that we get are the x-coordinates of the points of inflection.

Step 1: We already have the second derivative from part (b), so all we have to do is set it equal to zero and solve for x:

$$24x^2 - 8 = 0 \quad x^2 = \frac{1}{3} \quad x = \pm\sqrt{\frac{1}{3}}$$

PROBLEM 3: Water is draining at the rate of $48\pi \, \text{ft}^3$ from a conical tank whose diameter at its base is 40 feet and whose height is 60 feet.

(a) Find an expression for the volume of water (in ft^3) in the tank in terms of its radius.

The formula for the volume of a cone is: $V = \frac{1}{3}\pi R^2 H$, where R is the radius of the cone, and H is the height. The ratio of the height of a cone to its radius is constant at any point on the edge of the cone, so we also know that $\frac{H}{R} = \frac{60}{20} = 3$ (Remember that the radius is half the diameter.). If we solve this for H and substitute, we get:

$$H = 3R$$
$$V = \frac{1}{3}\pi R^2 (3R) = \pi R^3$$

(b) At what rate (in ft/sec) is the radius of the water in the tank shrinking when the radius is 16 feet?

Step 1: This is a Related Rates question. We now have a formula for the volume of the cone in terms of its radius, so if we differentiate it in terms of t we should be able to solve for the rate of change of the radius $\frac{dR}{dt}$.

We are given that the rate of change of the volume and the radius are, respectively:

$\frac{dV}{dt} = 48\pi$ and $R = 16$.

Differentiating the formula for the volume, we get: $\dfrac{dV}{dt} = 3\pi R^2 \dfrac{dR}{dt}$

Now we Plug In and get: $48\pi = 3\pi 16^2 \dfrac{dR}{dt}$. Finally, if we solve for $\dfrac{dR}{dt}$, we get:

$\dfrac{dR}{dt} = \dfrac{1}{16}$ ft/sec.

(c) How fast (in ft/sec) is the height of the water in the tank dropping at the instant that the radius is 16 feet?

Step 1: This is the same idea as the previous problem, except that we want to solve for $\dfrac{dH}{dt}$. In order to do this, we need to go back to our ratio of height to radius and solve it for the radius: $\dfrac{H}{R} = 3$ or $\dfrac{H}{3} = R$

Substituting for R in the original equation, we get: $V = \dfrac{1}{3}\pi \left(\dfrac{H}{3}\right)^2 H = \dfrac{\pi H^3}{27}$

Step 2: Now we need to know what H is when R is 16. Using our ratio:

$H = 3(16) = 48$.

Step 3: Now if we differentiate we get: $\dfrac{dV}{dt} = \dfrac{\pi H^2}{9} \dfrac{dH}{dt}$.

Now we Plug In and solve:

$$48\pi = \dfrac{\pi(48)^2}{9} \dfrac{dH}{dt}$$

$$\dfrac{dH}{dt} = \dfrac{3}{16}$$

One should also note that, because $H = 3R$, $\dfrac{dH}{dt} = 3\dfrac{dR}{dt}$ in part 2, we merely had to multiply it by 3 to find the answer for part 3.

PROBLEM 4. Let f be the function given by $f(x) = e^{-4x^2}$.

(a) Find the first four nonzero terms and the general term of the power series for $f(x)$ about $x = 0$.

Step 1: If we want to find the power series we need to do the Taylor Series expansion for $f(x)$. The formula for a Taylor Series about the point $x = a$ is:

$$f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \ldots + f^{(n)}(a)\frac{(x-a)^n}{n!} + \ldots$$

Here, $a = 0$ and $f(x) = e^{-4x^2}$. You should have memorized the Taylor Series expansion $e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{3!} + \ldots + \frac{u^n}{n!} + \ldots e^x$, so you don't have to do all of the work here. You can just substitute

$-4x^2$ for u: $e^{-4x^2} = 1 + \left(-4x^2\right) + \frac{\left(-4x^2\right)^2}{2} + \frac{\left(-4x^2\right)^3}{3!} = 1 - 4x^2 + 8x^4 - \frac{32x^6}{3}$

Step 2: The general term is: $\dfrac{(-1)^n 2^{2n} x^{2n}}{n!}$

(b) Find the interval of convergence of the power series for $f(x)$ about $x = 0$. Show the analysis that leads to your conclusion.

Step 1: The series for e^{-u^2} converges for $-\infty < u < \infty$. So the series for $f(x) = e^{-4x^2}$ converges if $-\infty < 2x < \infty$. Thus, it converges if $-\infty < x < \infty$.

(c) Use term-by-term differentiation to show that $f'(x) = -8xe^{-4x^2}$

Step 1: First we differentiate the individual terms of the power series:

$$f(x) = 1 - 4x^2 + 8x^4 - \frac{32x^6}{3} + \ldots + \frac{(-1)^n 2^{2n} x^{2n}}{n!}$$

$$f'(x) = 0 - 8x + 32x^3 - 64x^5 + \ldots + \frac{(-1)^n 2^{2n}(2n)x^{2n-1}}{n!}$$

$$= -8x + 32x^3 - 64x^5 + \ldots + \frac{(-1)^n 2^{2n+1} x^{2n-1}}{(n-1)!}$$

Step 2: Next, we multiply $-8x$ by the individual terms of the power series and compare to the result of step 1.

$$-8xf(x) = 1(-8x) - 4x^2(-8x) + 8x^4(-8x) - \frac{32x^6}{3}(-8x) + \ldots + \frac{(-1)^n 2^{2n} x^{2n}}{n!}(-8x)$$

$$= -8x + 32x^3 - 64x^5 + \ldots + \frac{(-1)^{n+1} 2^{2n+3} x^{2n+1}}{n!}$$

Step 3: If we substitute $n - 1$ for n, we can make the general terms match exactly.

$$\frac{(-1)^{(n-1)+1}2^{2(n-1)+3}x^{2(n-1)+1}}{(n-1)!} = \frac{(-1)^n 2^{2n+1}x^{2n-1}}{(n-1)!}$$

PROBLEM 5. Let R be the region enclosed by the graphs of $y = 2\ln x$ and $y = \dfrac{x}{2}$, and the lines $x = 2$ and $x = 8$.
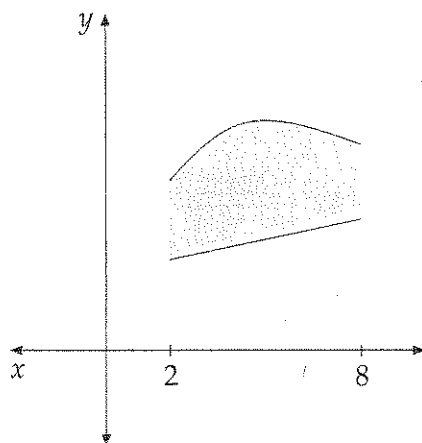
(a) Find the area of R.

Step 1: If there are two curves, $f(x)$ and $g(x)$, where $f(x)$ is always above $g(x)$, on the interval $[a,b]$, then the area of the region between the two curves is found by:

$$\int_a^b \big(f(x) - g(x)\big)dx$$

In order to determine whether one of the curves is above the other, we can graph them on the calculator.

The graph looks like this:

As we can see, the graph of $y = 2\ln x$ is above $y = \dfrac{x}{2}$ on the entire interval, so all we have to do is evaluate the integral $\displaystyle\int_2^8 \left(2\ln x - \frac{x}{2}\right)dx =$

Step 2: We can do the integration one of two ways—on the calculator or analytically.

Calculator: You should get 3.498

Analytically: $\int_2^8 \left(2\ln x - \frac{x}{2}\right) dx = 2\int_2^8 \ln x\,dx - \frac{1}{2}\int_2^8 x\,dx =$

$$2(x\ln x - x)\Big|_2^8 - \frac{1}{2}\left(\frac{x^2}{2}\right)\Big|_2^8 = 18.498 - 15 = 3.498$$

By the way, you should have memorized $\int \ln x\,dx = x\ln x - x$, or you can do it as one of the basic Integration By Parts integrals.

(b) Set up, but <u>do not integrate</u>, an integral expression, in terms of a single variable, for the volume of the solid generated when R is revolved about the <u>x-axis</u>.

Step 1: If there are two curves, $f(x)$ and $g(x)$, where $f(x)$ is always above $g(x)$, on the interval $[a, b]$, then the volume of the solid generated when the region is revolved about the x-axis is found by using the method of washers:

$$\pi \int_a^b [f(x)]^2 - [g(x)]^2\,dx$$

Here, we already know that $f(x)$ is above $g(x)$ on the interval, so the integral we

need to evaluate is: $\pi \int_2^8 [2\ln x]^2 - \left[\frac{x}{2}\right]^2 dx$

(c) Set up, but <u>do not integrate</u>, an integral expression, in terms of a single variable, for the volume of the solid generated when R is revolved about the line $x = -1$.

Step 1: Now we have to revolve the area around a <u>vertical</u> axis. If there are two curves, $f(x)$ and $g(x)$, where $f(x)$ is always above $g(x)$, on the interval $[a, b]$, then the volume of the solid generated when the region is revolved about the y-axis is found by using the method of shells:

$$2\pi \int_a^b x[f(x) - g(x)]\,dx$$

When we are rotating around a vertical axis, we use the same formula as when we rotate around the y-axis, but we have to account for the shift away from $x = 0$. Here we have a curve that is 1 unit farther away from the line $x = -1$ than it is from the y-axis, so we add 1 to the radius of the shell (For a more detailed explanation of shifting axes, see the unit on Finding the Volume of a Solid of Revolution). This gives us the equation:

$$2\pi \int_2^8 (x+1)\left[2\ln x - \frac{x}{2}\right] dx$$

PROBLEM 6. Let f and g be functions that are differentiable throughout their domains and that have the following properties:

(i) $f(x+y) = f(x)g(y) + g(x)f(y)$

(ii) $\lim\limits_{a \to 0} f(a) = 0$

(iii) $\lim\limits_{h \to 0} \dfrac{g(h)-1}{h} = 0$

(iv) $f'(0) = 1$

(a) Use L'Hopital's rule to show that $\lim\limits_{a \to 0} \dfrac{f(a)}{a} = 1$.

Step 1: L'Hopital's rule states that if a function is of the indeterminate form $\dfrac{0}{0}$, then the limit of the derivatives of the numerator is the same as the limit of the original quotient. In other words, if we differentiate the top and bottom of the quotient, we will get the limit. First, we have to show that the quotient gives us an indeterminate form.

$\lim\limits_{a \to 0} \dfrac{f(a)}{a} = \dfrac{0}{0}$ (using property (ii)). Next, we differentiate the top and bottom and

take the limit: $\lim\limits_{a \to 0} \dfrac{f(a)}{a} = \lim\limits_{a \to 0} \dfrac{f'(a)}{1} = f'(0) = 1$ (using property (iv)).

(b) Use the definition of the derivative to show that $f'(x) = g(x)$.

Step 1: The definition of the derivative states: $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h)-f(x)}{h}$

If we substitute into the formula, we get:

$f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h \to 0} \dfrac{f(x)g(h)+g(x)f(h)-f(x)}{h}$ (using property (i))

$$= \lim_{h \to 0} \frac{[g(h)-1]f(x)+g(x)f(h)}{h} = \lim_{h \to 0} \frac{[g(h)-1]f(x)}{h} + \frac{g(x)f(h)}{h}$$

$$\lim_{h \to 0} f(x)\frac{[g(h)-1]}{h} + g(x)\frac{f(h)}{h} = [f(x)](0) + g(x)(1) = g(x) \text{ (using property (iii))}$$

(c) Find $\int \frac{g(x)}{f(x)} dx$.

Step 1: Using u-substitution, let $u = f(x)$ and $du = f'(x) = g(x)$. Then we have:

$$\int \frac{g(x)}{f(x)} dx = \int \frac{du}{u} = \ln|u| + C = \ln|f(x)| + C$$